

# The thermodynamic limit for fractional exclusion statistics

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**Abstract.** I discuss Haldane's concept of generalised exclusion statistics (Phys. Rev. Lett. **67**, 937, 1991) and I show that it leads to inconsistencies in the calculation of the particle distribution that maximizes the partition function. These inconsistencies appear when mutual exclusion statistics is manifested between different subspecies of particles in the system. In order to eliminate these inconsistencies, I introduce new mutual exclusion statistics parameters, which are proportional to the dimension of the Hilbert sub-space on which they act. These new definitions lead to properly defined particle distributions and thermodynamic properties. In another paper (arXiv:0710.0728) I show that fractional exclusion statistics manifested in general systems with interaction have these, physically consistent, statistics parameters.

## 1. Introduction

Haldane's concept of fractional exclusion statistics (FES) [1] have been applied to the study of many types of physical systems, revealing interesting properties. For example it has been applied to strongly interacting systems, such as Tomonaga-Luttinger model [2, 3, 4, 5], Colagero-Sutherland model [6, 7, 8, 9, 10], fractional quantum Hall effect [11, 12, 13], or to interacting particles in one or two-dimensional systems, described in the mean field approximation [14, 15, 16, 17]. The statistical properties of FES systems have been calculated mainly by Isakov [18] and Wu [19], while Iguchi extended the Fermi liquid model to the model of a FES liquid [20, 21]; the microscopic reason for the manifestation of FES have also been discussed by several authors [14, 15, 10, 12, 13, 17, 22].

Although the concept received so much attention and has been applied in general to many types of systems, I will show here that when mutual exclusion statistics is manifested between different subspecies of particles in the system, FES leads to thermodynamic inconsistencies. I will also show that these inconsistencies can be corrected by a redefinition of the exclusion statistics parameters.

In a related paper I showed that fractional exclusion statistics appears in general systems of interacting particles and the statistics parameters indeed obey the rules conjectured here [23].

## 2. Thermodynamic inconsistencies in FES

In this section I will prove using two model systems, that in FES systems the equilibrium particle populations are ambiguously defined, if *mutual* statistics parameters are not zero. For this, I will recalculate the partition function and the most probable particle distribution in a FES system, following the procedure used by Wu in Ref. [19].

Haldane defined the fractional exclusion statistics as acting on Hilbert spaces of finite dimensions [1]. If we have only one such a space, in which we put  $N$  ideal bosons or fermions, then the number of microscopic configurations we have in the system is  $W_b = (G + N - 1)!/[N!(G - 1)!]$  (for bosons) or  $W_f = G!/[N!(G - N)!]$  (for fermions). Fractional exclusion statistics of parameter  $\alpha$  is an interpolation between these two cases and the number of configurations is  $W = [G + (N - 1)(1 - \alpha)]!/\{N![G - \alpha N - (1 - \alpha)]\}$ —we say that the addition of  $\delta N$  particles in the system reduces the number of available states in the system by  $\alpha \delta N$  [1, 19].

Now let us generalize the problem to the case when we have more than one Hilbert space. The spaces are  $\mathcal{H}_0, \mathcal{H}_1, \dots$ , of dimensions  $G_0, G_1, \dots$ , and which contain  $N_0, N_1, \dots$ , particles. In this case we have the FES parameters  $\alpha_{ij}$ , with  $i, j = 0, 1, \dots$ . Mutual exclusion statistics is manifested between the spaces  $\mathcal{H}_i$  and  $\mathcal{H}_j$  ( $i \neq j$ ) if  $\alpha_{ij} \neq 0$ —we say that the addition of  $\delta N_j$  particles in the space  $\mathcal{H}_j$  changes the number of available states in the space  $\mathcal{H}_i$  by  $-\alpha_{ij} \delta N_j$ . With these notations, the total number of configurations

is [19]

$$W = \prod_i \frac{[G_i + N_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})]!}{N_i! [G_i - 1 - \sum_j \alpha_{ij}(N_j - \delta_{ij})]!}. \quad (1)$$

Having the number of microscopic configurations (1), if we assign the energy  $\epsilon_i$  and the chemical potential  $\mu_i$  to the states in the space  $i$ , we can calculate the grandcanonical partition function,  $\mathcal{Z}$  [19],

$$\mathcal{Z} = \sum_{\{N_i\}} W(\{N_i\}) \exp \left[ \sum_i \beta N_i (\mu_i - \epsilon_i) \right], \quad (2)$$

and the total energy of the system in the given configuration,  $E = \sum_i N_i \epsilon_i$ —we use the notation  $\beta = 1/k_B T$ , where  $T$  is the temperature of the system.

The most probable configuration,  $\{N_i\}$ , is obtained by maximizing  $\mathcal{Z}$  with respect to the set  $\{N_i\}$ . If we introduce the notations  $n_i \equiv N_i/G_i$  and  $\beta_{ij} \equiv \alpha_{ij}G_j/G_i$ , and assume that for each  $i$  both,  $G_i$  and  $N_i$  are sufficiently large, so that we can use the Stirling approximation [ $\ln G_i! \approx G_i \ln(G_i/e)$  and  $\ln N_i! \approx N_i \ln(N_i/e)$ ] the maximization procedure gives us the system of equations,

$$n_i e^{\beta(\epsilon_i - \mu_i)} = \left[ 1 + \sum_k (\delta_{ik} - \beta_{ik}) n_k \right] \prod_j \left[ \frac{1 - \sum_k \beta_{jk} n_k}{1 + \sum_k (\delta_{jk} - \beta_{jk}) n_k} \right]^{\alpha_{ji}} \quad (3)$$

The system (3) is solved easier if we denote  $w_i \equiv n_i^{-1} - \sum_k \beta_{ik} n_k / n_i$ . Using this notations (3) becomes

$$(1 + w_i) \prod_j \left( \frac{w_j}{1 + w_j} \right)^{\alpha_{ji}} = e^{\beta(\epsilon_i - \mu_i)} \quad (4)$$

and  $n_i$ s can be calculated from the new system,

$$\sum_j (\delta_{ij} w_j + \beta_{ij}) n_j = 1. \quad (5)$$

If the spaces  $i$  are fixed, Eq. (5) can be solved and the populations  $n_i$  can be determined. But in general, the spaces  $\mathcal{H}_i$  can be changed by dividing or combining them and, if the system has a proper thermodynamic behavior, such changes should not affect the thermodynamic results. Unfortunately this is not the case and I will show now by simple examples that the particle distribution that maximizes the partition function depends on our choice of subspaces  $\mathcal{H}_i$ .

Let us assume that we have only two finite dimensional Hilbert spaces,  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , of dimensions  $G_0$  and  $G_1$ , and the FES parameters  $\alpha_{00}$ ,  $\alpha_{11}$ ,  $\alpha_{01}$ , and  $\alpha_{10}$ . Then Eqs. (3) and (4) reduce to systems of two equations and two unknowns:

$$\begin{cases} (1 + w_0) \left( \frac{w_0}{1 + w_0} \right)^{\alpha_{00}} \left( \frac{w_1}{1 + w_1} \right)^{\alpha_{10}} = e^{\beta(\epsilon_0 - \mu_0)} \\ (1 + w_1) \left( \frac{w_0}{1 + w_0} \right)^{\alpha_{01}} \left( \frac{w_1}{1 + w_1} \right)^{\alpha_{11}} = e^{\beta(\epsilon_1 - \mu_1)} \end{cases} \quad (6)$$

and

$$\begin{cases} (w_0 + \alpha_{00})n_0 + \alpha_{01}\frac{G_1}{G_0}n_1 = 1 \\ \alpha_{10}\frac{G_0}{G_1}n_0 + (w_1 + \alpha_{11})n_1 = 1 \end{cases} \quad (7)$$

respectively. Now let us further assume that the space  $\mathcal{H}_1$  was obtained as a union of the two smaller, disjoint subspaces,  $\mathcal{H}_{1a}$  and  $\mathcal{H}_{1b}$ , of dimensions  $G_{1a}$  and  $G_{1b}$ , respectively. Then, instead of two Hilbert spaces and four exclusion statistics parameters that we started with, we can also describe the system as consisting of three subspaces and nine statistics parameters. From these nine,  $\alpha_{00}$  remains unchanged, whereas  $\alpha_{01a}$  and  $\alpha_{01b}$  should be identical to  $\alpha_{01}$ . A natural choice for the other six parameters is  $\alpha_{1a1a} = \alpha_{1b1b} \equiv \alpha_{11}$ ,  $\alpha_{1b0} = \alpha_{1a0} = \alpha_{10}$ , and  $\alpha_{1a1b} = \alpha_{1b1a} = 0$ . Obviously, we also have  $\epsilon_{1a} = \epsilon_{1b} = \epsilon_1$  and  $\mu_{1a} = \mu_{1b} = \mu_1$ . In the new configuration, the systems (4) and (5) become

$$\begin{cases} (1 + w'_0) \left( \frac{w'_0}{1+w'_0} \right)^{\alpha_{00}} \left( \frac{w_{1a}}{1+w_{1a}} \right)^{\alpha_{10}} \left( \frac{w_{1b}}{1+w_{1b}} \right)^{\alpha_{10}} = e^{\beta(\epsilon_0 - \mu_0)} \\ (1 + w_{1a}) \left( \frac{w'_0}{1+w'_0} \right)^{\alpha_{01}} \left( \frac{w_{1a}}{1+w_{1a}} \right)^{\alpha_{11}} = e^{\beta(\epsilon_1 - \mu_1)} \\ (1 + w_{1b}) \left( \frac{w'_0}{1+w'_0} \right)^{\alpha_{01}} \left( \frac{w_{1b}}{1+w_{1b}} \right)^{\alpha_{11}} = e^{\beta(\epsilon_1 - \mu_1)} \end{cases} \quad (8)$$

and

$$\begin{cases} (w'_0 + \alpha_{00})n_0 + \alpha_{01}\frac{G_{1a}}{G_0}n_{1a} + \alpha_{01}\frac{G_{1b}}{G_0}n_{1b} = 1 \\ \alpha_{10}\frac{G_0}{G_{1a}}n_0 + (w_{1a} + \alpha_{11})n_{1a} = 1 \\ \alpha_{10}\frac{G_0}{G_{1b}}n_0 + (w_{1b} + \alpha_{11})n_{1b} = 1 \end{cases} \quad (9)$$

The systems (6) and (7) should admit the same physical solution as the systems (8) and (9), which is  $n_{1a} = n_{1b} = n_1$ . If we also take, for the simplicity of the calculations,  $G_{1a} = G_{1b} = G_1/2$ , then from the last two equations of (9) we get  $w_{1a} = w_{1b}$  and

$$\alpha_{10}n_0G_0/G_1 + (w_{1a} - w_1)n_1 = 0. \quad (10)$$

Using  $w_{1a} = w_{1b} \equiv w'_1$  into (8), we obtain the system

$$\begin{cases} (1 + w'_0) \left( \frac{w'_0}{1+w'_0} \right)^{\alpha_{00}} \left( \frac{w'_1}{1+w'_1} \right)^{2\alpha_{10}} = e^{\beta(\epsilon_0 - \mu_0)} \\ (1 + w'_1) \left( \frac{w'_0}{1+w'_0} \right)^{\alpha_{01}} \left( \frac{w'_1}{1+w'_1} \right)^{\alpha_{11}} = e^{\beta(\epsilon_1 - \mu_1)} \end{cases} \quad (11)$$

Since the systems (6) and (11) are not identical, Eqs. (11) and (10) (eventually combined also with 6) give three equations for the two unknowns,  $w'_0$  and  $w'_1$ . The system of equations is overdetermined and does not have solutions, unless the statistics parameters have some special values. Such special values are e.g.  $\alpha_{10} = \alpha_{01} = 0$  ( $\alpha_{01} = 0$  would be necessary if we would split the space  $\mathcal{H}_0$  instead of  $\mathcal{H}_1$ ).

### 2.1. Large systems

I will show now that the thermodynamic inconsistency of FES happens also in systems of infinite, quasi-continuous spectra. For this let us assign to the single-particle states the sets of quantum numbers,  $\mathbf{k}$ —the set  $\mathbf{k}$  does not necessary consist of wave-vector components, as it usually does, but it may contain any quantum numbers. If the system is large enough, we say that each of the subspaces  $\mathcal{H}_i$  contains the set of single particle states  $\{\mathbf{k}\}_i$ , which “fill-up” a volume  $V_i$  in the space of the quantum numbers and we define the density of states,  $\sigma_{\mathbf{k}}$ , so that  $G_i = \int_{V_i} \sigma_{\mathbf{k}} d\mathbf{k}$  and  $N_i = \int_{V_i} n_{\mathbf{k}} \sigma_{\mathbf{k}} d\mathbf{k}$ . Then the parameters  $\alpha_{ij}$  depend on the quantum numbers  $\{\mathbf{k}\}_i$  and  $\{\mathbf{k}\}_j$ , so we shall change the subscripts and write in general, e.g.  $\alpha_{\mathbf{k}\mathbf{k}'}$ . In these new notations, the system (4) becomes

$$(1 + w_{\mathbf{k}}) \prod_{\mathbf{k}'} \left( \frac{w_{\mathbf{k}'}}{1 + w_{\mathbf{k}'}} \right)^{\alpha_{\mathbf{k}'\mathbf{k}}} = e^{\beta(\epsilon_{\mathbf{k}} - \mu_{\mathbf{k}})} \quad (12)$$

for any division of the total Hilbert space of the system into elementary volumes  $V_i$ , as long as  $G_i$  and  $N_i$  (for any  $i$ ) are large enough, so that we can apply the Stirling formula and the maximization procedure presented above. Moreover, if we can choose all the volumes  $V_i$  so that they are small enough to use the approximation  $G_i = \sigma_{\mathbf{k}} V_i$  and  $\alpha_{ij} = \alpha_{\mathbf{k}\mathbf{k}'}$  for any  $\mathbf{k}$  in  $V_i$  and  $\mathbf{k}'$  in  $V_j$ , Eq. (5) becomes

$$\sum_{\mathbf{k}'} \left[ \delta_{\mathbf{k}\mathbf{k}'} w_{\mathbf{k}'} + \alpha_{\mathbf{k}\mathbf{k}'} \frac{\sigma_{\mathbf{k}'}}{\sigma_{\mathbf{k}}} \cdot \frac{V_{\mathbf{k}'}}{V_{\mathbf{k}}} \right] n_{\mathbf{k}'} = 1, \quad (13)$$

where we also changed the notation  $V_i$  into  $V_{\mathbf{k}}$  and  $V_j$  into  $V_{\mathbf{k}'}$ . Now we observe directly that while Eqs. (12) contain only intensive parameters (i.e. which do not depend on the volumes  $V_{\mathbf{k}}$  involved), the values of the equilibrium particle populations (13) depend on our arbitrary choice of volumes, so the thermodynamic quantities, which are the populations  $n_{\mathbf{k}}$ , are not well defined.

### 3. Correction of the parameters

The reason for which the thermodynamics of FES systems is ambiguous is obvious from the beginning. If we look for example at Eq. (3) and we imagine that we reduce the dimension of one of the spaces, say of space  $i$ , by half, than since all the parameters  $\beta_{ij}$  are proportional to  $G_j/G_i$ , this could cause a significant redistribution of all the occupation numbers (as we proved in concrete cases above). Since the population numbers should be intensive parameters, it means that the system could not have thermodynamic limit.

Fortunately, as it was easy to observe the anomaly of FES, it is as easy to correct it. I shall prove below that if the “trouble-making” mutual exclusion statistics parameters are proportional to the space on which they act, i.e.  $\alpha_{ij} \propto G_i$ , then the system admits a thermodynamic limit. In Ref. [23] I analyzed the exclusion statistics that appears in general interacting systems and I showed that in the limit of a quasi-continuous density of states the mutual exclusion statistics parameters indeed satisfy this property. To

differentiate these new parameters from the ones used above, I shall denote them by  $\tilde{\alpha}_{\mathbf{k}\mathbf{k}'}$ . Therefore, the qualitatively new feature of the parameters  $\tilde{\alpha}_{\mathbf{k}\mathbf{k}'}$  is that for  $\mathbf{k} \neq \mathbf{k}'$ ,  $\tilde{\alpha}_{\mathbf{k}\mathbf{k}'}$  is not a simple number, but *it is proportional to the dimension of the subspace  $V_{\mathbf{k}}$* :  $\tilde{\alpha}_{\mathbf{k}\mathbf{k}'} = G_{V_{\mathbf{k}}} \alpha_{\mathbf{k}\mathbf{k}'} \equiv \sigma_{\mathbf{k}} V_{\mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'}$ . If  $\mathbf{k} = \mathbf{k}'$ , the statistics parameters are similar to the ones studied so far and we write  $\tilde{\alpha}_{\mathbf{k}\mathbf{k}} = \alpha_{\mathbf{k}\mathbf{k}}$ .

Let us now try to find the equilibrium configuration with these new statistics parameters and see if this has a thermodynamic limit. To do this, we first have to write down the new number of configurations (1):

$$W = \prod_{\mathbf{k}} \frac{[G_{V_{\mathbf{k}}} + (1 - \alpha_{\mathbf{k}\mathbf{k}})(N_{V_{\mathbf{k}}} - 1) - G_{V_{\mathbf{k}}} \sum_{\mathbf{k}' \neq \mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'} N_{V_{\mathbf{k}'}}]!}{N_i! [G_{V_{\mathbf{k}}} - 1 - \alpha_{\mathbf{k}\mathbf{k}}(N_{V_{\mathbf{k}}} - 1) - G_{V_{\mathbf{k}}} \sum_{\mathbf{k}' \neq \mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'} N_{V_{\mathbf{k}'}}]!}. \quad (14)$$

But we notice already that for large enough volumes  $V_{\mathbf{k}}$ , both,  $G_{V_{\mathbf{k}}}$  and  $N_{V_{\mathbf{k}}}$  are large, therefore the summations  $G_{V_{\mathbf{k}}} \sum_{\mathbf{k}' \neq \mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'} N_{V_{\mathbf{k}'}}$ , containing terms bilinear in these large quantities, must be much larger than  $G_{V_{\mathbf{k}}}$  and  $N_{V_{\mathbf{k}}}$ . In conclusion both square brackets in (14) should be negative. This is not admissible. [We note in passing that also in Haldane-Wu formalism (1) [1, 19] negative terms may appear in the calculation of the number of configurations for inappropriate choice of subspaces, even though in the summations there are only terms linear in  $N_i$ .]

To solve the problem we have to regard the exclusion statistics as acting on the quasiparticle added as a perturbation to the equilibrium quasiparticle distribution. Therefore we assume that on top of the equilibrium distribution of particles, say  $N_{V_{\mathbf{k}}}$ , we add a small perturbation,  $\delta N_{V_{\mathbf{k}}}$ , which changes the number of configurations into

$$\begin{aligned} W &= \prod_{\mathbf{k}} \frac{[\tilde{G}_{V_{\mathbf{k}}} + N_{V_{\mathbf{k}}} + \delta N_{V_{\mathbf{k}}} - 1 - \sum_{\mathbf{k}' \neq \mathbf{k}} \tilde{\alpha}_{\mathbf{k}\mathbf{k}'} \delta N_{V_{\mathbf{k}'}}]!}{(N_{V_{\mathbf{k}}} + \delta N_{V_{\mathbf{k}}})! (\tilde{G}_{V_{\mathbf{k}}} - 1 - \sum_{\mathbf{k}' \neq \mathbf{k}} \tilde{\alpha}_{\mathbf{k}\mathbf{k}'} \delta N_{V_{\mathbf{k}'}})!} \\ &= \prod_{\mathbf{k}} \frac{[\tilde{G}_{V_{\mathbf{k}}} + N_{V_{\mathbf{k}}} + (1 - \alpha_{\mathbf{k}\mathbf{k}}) \delta N_{V_{\mathbf{k}}} - 1 - \tilde{G}_{V_{\mathbf{k}}} \sum_{\mathbf{k}' \neq \mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'} \delta N_{V_{\mathbf{k}'}}]!}{(N_{V_{\mathbf{k}}} + \delta N_{V_{\mathbf{k}}})! (\tilde{G}_{V_{\mathbf{k}}} - \alpha_{\mathbf{k}\mathbf{k}} \delta N_{V_{\mathbf{k}}} - 1 - \tilde{G}_{V_{\mathbf{k}}} \sum_{\mathbf{k}' \neq \mathbf{k}} \alpha_{\mathbf{k}\mathbf{k}'} \delta N_{V_{\mathbf{k}'}})!}. \end{aligned} \quad (15)$$

Notice that now  $\tilde{G}_{V_{\mathbf{k}}}$  is the Bose dimension of the Hilbert space, i.e. the number of available states, and not its real dimension. So, unless we have an ideal Bose system, the ratio  $N_{V_{\mathbf{k}}}/\tilde{G}_{V_{\mathbf{k}}}$ , denoted by  $\tilde{n}_{\mathbf{k}}$ , is actually the ratio of the number of particles to the number of available states (or holes). In terms of  $G_{\mathbf{k}}$  of equation (1) or (14),  $\tilde{G}_{V_{\mathbf{k}}} = G_{\mathbf{k}} - \alpha N_{\mathbf{k}}$ . The equilibrium distribution is then obtained by imposing that the partition function is stationary with respect to the perturbations  $\delta N_{V_{\mathbf{k}}}$ . If we introduce the notations,  $\delta \tilde{n}_{\mathbf{k}} \equiv \delta N_{V_{\mathbf{k}}}/\tilde{G}_{V_{\mathbf{k}}}$ , using the Stirling approximation we write the logarithm of  $W$  as

$$\begin{aligned} \ln W &= \sum_{\mathbf{k}} \tilde{G}_{V_{\mathbf{k}}} \left\{ \left[ 1 + \tilde{n}_{\mathbf{k}} + (1 - \alpha_{\mathbf{k}\mathbf{k}}) \delta \tilde{n}_{\mathbf{k}} - \sum_{\mathbf{k}' (\neq \mathbf{k})} \alpha_{\mathbf{k}\mathbf{k}'} \tilde{G}_{V_{\mathbf{k}'}} \delta \tilde{n}_{\mathbf{k}'} \right] \right. \\ &\quad \times \ln \left[ 1 + \tilde{n}_{\mathbf{k}} + (1 - \alpha_{\mathbf{k}\mathbf{k}}) \delta \tilde{n}_{\mathbf{k}} - \sum_{\mathbf{k}' (\neq \mathbf{k})} \alpha_{\mathbf{k}\mathbf{k}'} \tilde{G}_{V_{\mathbf{k}'}} \delta \tilde{n}_{\mathbf{k}'} \right] \\ &\quad \left. - (\tilde{n}_{\mathbf{k}} + \delta \tilde{n}_{\mathbf{k}}) \ln(\tilde{n}_{\mathbf{k}} + \delta \tilde{n}_{\mathbf{k}}) - \left( 1 - \alpha_{\mathbf{k}\mathbf{k}} \delta \tilde{n}_{\mathbf{k}} - 1 - \sum_{\mathbf{k}' (\neq \mathbf{k})} \alpha_{\mathbf{k}\mathbf{k}'} \tilde{G}_{V_{\mathbf{k}'}} \delta \tilde{n}_{\mathbf{k}'} \right) \right\} \end{aligned}$$

$$\times \ln \left( 1 - \alpha_{\mathbf{k}\mathbf{k}} \delta \tilde{n}_{\mathbf{k}} - 1 - \sum_{\mathbf{k}' (\neq \mathbf{k})} \alpha_{\mathbf{k}\mathbf{k}'} \tilde{G}_{\mathbf{k}'} \delta \tilde{n}_{\mathbf{k}'} \right) \Bigg\} \quad (16)$$

Adding  $\beta \sum_{\mathbf{k}} \tilde{G}_{V_{\mathbf{k}}} \tilde{n}_{\mathbf{k}} (\mu_{\mathbf{k}} - \epsilon_{\mathbf{k}})$  to (16), we obtain the logarithm of the partition function, which, if maximized, gives

$$\begin{aligned} 0 &= \frac{\partial}{\partial n_{\mathbf{k}}} \left[ \ln W + \beta \sum_{\mathbf{k}'} \tilde{G}_{V_{\mathbf{k}'}} \tilde{n}_{\mathbf{k}'} (\mu_{\mathbf{k}'} - \epsilon_{\mathbf{k}'}) \right] \\ &= \tilde{G}_{V_{\mathbf{k}}} \left\{ \beta (\mu_{\mathbf{k}} - \epsilon_{\mathbf{k}}) - \ln \tilde{n}_{\mathbf{k}} + (1 - \alpha_{\mathbf{k}\mathbf{k}}) \ln(1 + \tilde{n}_{\mathbf{k}}) - \sum_{\mathbf{k}' (\neq \mathbf{k})} \alpha_{\mathbf{k}'\mathbf{k}} \tilde{G}_{V_{\mathbf{k}'}} \ln(1 + \tilde{n}_{\mathbf{k}'}) \right\} \end{aligned} \quad (17)$$

We observe now that if  $\alpha_{\mathbf{k}\mathbf{k}'} = 0$ , for any  $\mathbf{k}$  and  $\mathbf{k}'$ , we obtain the Bose distribution,  $\tilde{n}_{\mathbf{k}} = n_{\mathbf{k}} = \{\exp[\beta(\epsilon_{\mathbf{k}} - \mu_{\mathbf{k}}) - 1]\}^{-1}$ , whereas if  $\alpha_{\mathbf{k}\mathbf{k}'} = 0$ , for any  $\mathbf{k} \neq \mathbf{k}'$ , and  $\alpha_{\mathbf{k}\mathbf{k}} = 1$  for any  $\mathbf{k}$ , we obtain  $\tilde{n}_{\mathbf{k}} = \exp[\beta(\mu_{\mathbf{k}} - \epsilon_{\mathbf{k}})]$ , which is the ratio of the number of particles to the number of holes in a Fermi system, as expected.

Now we can finally extract from (17) a self-consistent equation for  $n_{\mathbf{k}}$ :

$$\beta(\mu_{\mathbf{k}} - \epsilon_{\mathbf{k}}) + \ln \frac{[1 + \tilde{n}_{\mathbf{k}}]^{1 - \alpha_{\mathbf{k}\mathbf{k}}}}{\tilde{n}_{\mathbf{k}}} = \sum_{\mathbf{k}' (\neq \mathbf{k})} G_{V_{\mathbf{k}'}} \ln[1 + \tilde{n}_{\mathbf{k}'}] \alpha_{\mathbf{k}'\mathbf{k}} \quad (18)$$

In the thermodynamic limit, when  $G_{V_{\mathbf{k}}} \ll \sum_{\mathbf{k}' (\neq \mathbf{k})} G_{V_{\mathbf{k}'}}$ , Eq. (18) gives a good, unambiguous particle distribution in the system. The summation to the right-hand side of Eq. (18) can be readily and consistently transformed into an intergral when the dimension of the system increases and we get the self-consistent integral equation for the particle distribution,

$$\beta(\mu_{\mathbf{k}} - \epsilon_{\mathbf{k}}) + \ln \frac{[1 + \tilde{n}_{\mathbf{k}}]^{1 - \alpha_{\mathbf{k}\mathbf{k}}}}{\tilde{n}_{\mathbf{k}}} = \int \sigma_{\mathbf{k}'} \ln[1 + \tilde{n}_{\mathbf{k}'}] \alpha_{\mathbf{k}'\mathbf{k}} d\mathbf{k}'. \quad (19)$$

In the end let us remark an identity. If all the mutual statistics parameters were zero, the model introduced here should be identical to the previous FES model (section 2) so so let us compare the results. First, if  $\alpha_{\mathbf{k}\mathbf{k}'} = 0$  for all  $\mathbf{k} \neq \mathbf{k}'$ , equation (18) reduces to

$$\frac{(1 + \tilde{n}_{\mathbf{k}})^{1 - \alpha_{\mathbf{k}\mathbf{k}}}}{\tilde{n}_{\mathbf{k}}} = e^{\beta(\epsilon_{\mathbf{k}} - \mu_{\mathbf{k}})}, \quad (20)$$

whereas equation (12) becomes

$$\frac{w_{\mathbf{k}}^{\alpha_{\mathbf{k}\mathbf{k}}}}{(1 + w_{\mathbf{k}})^{\alpha_{\mathbf{k}\mathbf{k}} - 1}} = e^{\beta(\epsilon_{\mathbf{k}} - \mu_{\mathbf{k}})} \quad (21)$$

Now one can check easily that equation (20) is identical to equation (21) if  $\tilde{n}_{\mathbf{k}} = w_{\mathbf{k}}^{-1}$ . But on the other hand, by definition,  $w_{\mathbf{k}}^{-1} = N_{\mathbf{k}} / (G_{\mathbf{k}} - \alpha_{\mathbf{k}\mathbf{k}} N_{\mathbf{k}}) = N_{\mathbf{k}} / \tilde{G}_{\mathbf{k}} \equiv \tilde{n}_{\mathbf{k}}$ . Therefore the results are indeed identical and we observe with this occasion that in FES systems with no mutual statistics, the quantity  $w_{\mathbf{k}}$  was the ration between the Bose dimension of the subspace  $V_{\mathbf{k}}$  and the number of particles in this subspace.

## 4. Conclusions

I showed that fractional exclusion statistics leads to ambiguous thermodynamic results if the mutual exclusion statistics parameters are not zero ( $\alpha_{ij} \neq 0$ , for  $i \neq j$ ). To correct this ambiguity and to obtain consistent thermodynamic results I introduced new mutual exclusion statistics parameters, which are proportional to the dimension of the subspace on which they act.

In a related publication [23] I proved that a gas of interacting particles can be described as a fractional exclusion statistics gas, with the exclusion statistics parameters having the properties obtained here.

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